5. NEW CHARACTERS FROM OLD

§5.1. Inducing from Quotient Groups

It is possible to construct characters of a group from those of subgroups or quotient groups. In this way we can proceed from smaller groups to larger ones. Let's begin with quotient groups.

Theorem 1: Suppose H is a normal subgroup of G and $\pi: G \rightarrow G/H$ maps g to gH. Suppose that

$$\rho: G/H \rightarrow GL(n, F)$$

is a representation of G/H. Then:

(1) ${}^G\rho = \pi\rho: G \rightarrow GL(n, F)$ is a representation of G.

(2) if ρ is irreducible, so is ${}^G\rho$.

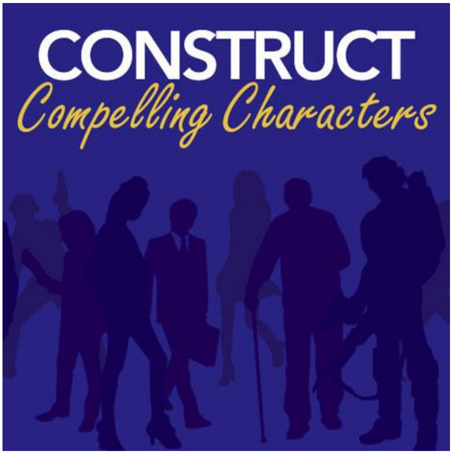
(3) if χ is the corresponding character of G then ${}^G\chi = \pi\chi$ is a character of G.

(4) if χ is irreducible then so is ${}^G\rho$.

Proof: (1) The product of two homomorphisms is a homomorphism.

(2), (3) and (4) are obvious. ✋☺

Example 1: If G = S_4 then H = {I, (12)(34), (13)(24), (14)(23)} is a normal subgroup of G, with G/H ≅ S_3. The complex character table for S_3 is:



class	Γ_1	Γ_2	Γ_3
size	1	3	2
χ_1	1	1	1
χ_2	1	–1	1
χ_3	2	0	–1

where Γ_1 = {I}, Γ_2 = {(12), (13), (23)} and Γ_3 = {(123), (132)}.

Now S_4 has 5 conjugacy classes, corresponding to the cycle structures I, (××), (×××), (××××), (××)(××). Let the corresponding conjugacy classes be Ω_1, Ω_2, Ω_3, Ω_4, Ω_5. Each Γ_i consists of cosets whose representatives come from one or more Ω_j. The correspondences are as follows:

$$\Gamma_1 \leftrightarrow \Omega_1 + \Omega_5,\ \Gamma_2 \leftrightarrow \Omega_2 + \Omega_4,\ \Gamma_3 \leftrightarrow \Omega_3$$

Hence we get 3 of the 5 irreducible characters of S_4 by inducing each of the irreducible characters of G/H.

class	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
size	1	6	8	6	3
${}^G\chi_1$	1	1	1	1	1
${}^G\chi_2$	1	–1	1	–1	1
${}^G\chi_3$	2	0	–1	0	2

§ 5.2. Character Tables of Direct Products

Theorem 2: Suppose the conjugacy classes of H are Γ_1, Γ_2, …, Γ_r with corresponding sizes a_1, a_2, …, a_r and suppose the conjugacy classes of K are Ω_1, Ω_2, …, Ω_s with corresponding sizes b_1, b_2, …, b_s.

Suppose the character tables for H, K are $\chi = (\chi_{ij})$ and $\Phi = (\Phi_{ij})$ respectively.

(1) The set of conjugacy classes of H × K is $\{\Gamma_i \times \Omega_j\}$.

(2) The character table for H × K is the $rs \times rs$ tensor product $(\chi_{ij}) \otimes (\Phi_{ij})$.

Proof: (1) This follows from the fact that if x, h ∈ H and y, k ∈ K then

$$(x, y)^{-1}(h, k)(x, y) = (x^{-1}hx, y^{-1}ky).$$

(2) Let $\rho: H \to M_m(F)$ and $\sigma: K \to M_n(F)$ be irreducible representations for H, K over a field F.

Then $(\rho_1 \otimes \rho_2): H \times K \to M_{mn}(F)$, defined by:

$$(\rho_1 \otimes \rho_2)(h, k) = \rho(h) \otimes \sigma(k),$$

is a representation of H × K.

Let $\chi(h) = tr[\rho(h)]$ and $\Phi(k) = tr[\sigma(k)]$, that is the value of the characters χ and Φ on h, k respectively.

The character of $\rho_1 \otimes \rho_2$ is the trace of:

$$\rho(h) \otimes \sigma(k) = tr[\rho(h)].tr[\sigma(k)] = \chi(h).\Phi(k).$$

Let this character of H × K be denoted by θ.

$$\langle \theta \mid \theta \rangle = \frac{1}{|H|.|K|} \sum_{h\in H, k\in K} |\chi(h)\Phi(k)|^2.$$

Here we don't weight the terms by the sizes of the conjugacy classes since we are summing over the elements and not the classes.

$$\text{So } \langle \theta \mid \theta \rangle = \frac{1}{|H|} \sum_{h\in H} |\chi(h)|^2 . \frac{1}{|K|} \sum_{k\in K} |\Phi(k)|^2.$$

Since χ and Φ are irreducible, both these sums are 1 and so $\langle \theta \mid \theta \rangle = 1$.
Hence θ is an irreducible representation of $H \times K$.

Now there are *r* possibilities for χ and *s* possibilities for Φ, giving *rs* possible irreducible representations for $H \times K$. This is how many conjugacy classes there are in $H \times K$ and hence the number of irreducible representations for $H \times K$.

But before we can say that we've found them all, we must check that they are distinct.

Suppose that χ' and Φ' are also irreducible representations and θ' is the representation of $H \times K$ where

$$\theta'(h, k) = \chi'(h).\Phi'(k).$$

$$\text{Then } \langle \theta \mid \theta' \rangle = \frac{1}{|H|.|K|} \sum_{h\in H, k\in K} \overline{\chi(h)\Phi(k)}\, \chi'(h)\Phi'(k)$$

$$= \frac{1}{|H|} \sum_{h\in H, k\in K} \overline{\chi(h)}\chi'(k) \cdot \frac{1}{|K|} \overline{\Phi(h)}\Phi'(k) .$$

If either $\chi \neq \chi'$ or $\Phi \neq \Phi'$ this will be zero. Hence the rs irreducible representations that we obtain in this way will be mutually orthogonal, and hence distinct.

It remains to obtain the characters of these representations. Let Γ_{ij} be the Cartesian product $\Gamma_i \times \Phi_j$. If we order the conjugacy classes of H × K as:

$$\Gamma_{11}, \Gamma_{12}, \ldots, \Gamma_{1s}, \Gamma_{21}, \Gamma_{22}, \ldots \Gamma_{2s}, \Gamma_{r1}, \Gamma_{r2}, \ldots, \Gamma_{rs}$$

the character table of H × K will be $(\chi_{ij}) \otimes (\Phi_{ij})$. ✋☺

Example 3: The character table for A_4 is

class	**I**	**(××)(××)**	**(×××)**	**(×××)**
size	**1**	**3**	**4**	**4**
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	–1	0	0
order	**1**	**2**	**3**	**3**

and for S_3 it is

class	I	(×××)	(××)
size	**1**	**2**	**3**
χ_1	1	1	1
χ_2	1	1	–1
χ_3	2	–1	0
order	**1**	**3**	**2**

Let *mn* denote the product of a cycle of length *m* by a cycle of length *n*. So 22 represents (××) and 3 represents (×××).

The character table for $A_4 \times S_3$ is:

class	(I,I)	(22,I)	(3,I)	(3,I)	(I,3)	(22,3)
size	**1**	**3**	**4**	**4**	**2**	**6**
χ_1	1	1	1	1	1	1
χ_2	1	1	ω	ω^2	1	1
χ_3	1	1	ω^2	ω	1	1
χ_4	3	–1	0	0	3	–1
χ_5	1	1	1	1	1	1
χ_6	1	1	ω	ω^2	1	1
χ_7	1	1	ω^2	ω	1	1
χ_8	3	–1	0	0	3	–1
χ_9	2	2	2	2	–1	–1
χ_{10}	2	2	2ω	$2\omega^2$	–1	–1
χ_{11}	2	2	$2\omega^2$	2ω	–1	–1
χ_{12}	6	–2	0	0	–3	1
order	**1**	**2**	**3**	**3**	**3**	**6**

class	(3,3)	(3,3)	(I,2)	(22,2)	(3,2)	(3,2)
size	8	8	3	9	12	12
χ_1	1	1	1	1	1	1
χ_2	ω	ω^2	1	1	ω	ω^2
χ_3	ω^2	ω	1	1	ω^2	ω
χ_4	0	0	3	–1	0	0
χ_5	1	1	–1	–1	–1	–1
χ_6	ω	ω^2	–1	–1	$-\omega$	$-\omega^2$
χ_7	ω^2	ω	–1	–1	$-\omega^2$	$-\omega$
χ_8	0	0	–3	1	0	0
χ_9	–1	–1	0	0	0	0
χ_{10}	$-\omega$	$-\omega^2$	0	0	0	0
χ_{11}	$-\omega^2$	$-\omega$	0	0	0	0
χ_{12}	0	0	0	0	0	0
order	3	3	2	2	6	6

§5.3. Inducing from Subgroups

Inducing up from quotient is of no use if we want to find the character table of a simple group. But simple groups have plenty of subgroups, and it’s possible to induce up from them. The big difference between the subgroup and quotient group situation is that when we induce up from an irreducible character of a quotient we always get an irreducible character of the larger group. But when we induce up from a subgroup we rarely get an irreducible character. Extra work is needed to split it into irreducible characters.

Suppose U, V are finite-dimensional vector spaces over a field F with bases

$\{u_1, u_2, \ldots, u_m\}$ and $\{v_1, v_2, \ldots , v_n\}$

respectively.

We denote the ordered pair (u_i, v_j) by $u_i \otimes v_j$ and define the **tensor product** $U \otimes V$ to be the space spanned by the $u_i \otimes v_j$. The dimension of $U \otimes V$ is the product of the dimensions of U and V.

Let $H \leq G$ and let $Y = \{y_1, y_2, \ldots , y_n\}$ be a right transversal of H in G (a set of representatives, one from each right coset of H in G). If F is a field then FY is the subspace of FG spanned by the elements of Y.

Let U be a finite dimensional vector space with a basis $\{u_1, u_2, \ldots , u_m\}$. Then $FY \otimes U$ will have:

$\{y_i \otimes u_j \mid i = 1, 2, \ldots , n \text{ and } j = 1, 2, \ldots , m\}$

as a basis. These $y_i \otimes u_j$ are simply ordered pairs, written a little differently to usual.

Let $\rho{:}H\rightarrow Aut_F(U)$ be a representation of H on the vector space U. The elements of $FY \otimes U$ will be linear combinations of the $y_i \otimes u_j$.

We can define a linear transformation from $FY \otimes U$ to itself by defining its effect on each basis vector.

For $g \in G$ define $(y_i \otimes u_j)(g\rho^G) = y_k \otimes u_j(y_i g y_k^{-1})\rho)$ where $y_i g \in Hy_k$. In other words y_k is the representative of the coset that $y_i g$ belongs to.

Note that $y_i g y_k^{-1} \in H$ so $(y_i g y_k^{-1})\rho$ is defined and $u_j(y_i g y_k^{-1})\rho \in U$ and so is a linear combination of the u_j.

Having defined $g\rho^G$ on each basis vector of $FY \otimes U$ we can extend it to the whole space by linearity. If ρ^G is the representation of G induced from a subgroup H, the corresponding character is denoted by $\boldsymbol{\chi}^{\mathbf{G}}$.

Example 2: Let $G = D_6 = \langle A, B \mid A^3 = B^2 = 1, BA = A^{-1}B\rangle$ and let $H = \{1, AB\}$.
Since H is a cyclic group of order 2 we can define a representation of degree 2 by mapping AB to a 2×2 matrix of order 2, with 1 mapping to the 2×2 identity matrix.
Define $(AB)\rho = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$. Let U be the set of row vectors (x, y) over $\mathbb{C}$.
Take the standard basis $e_1 = (1, 0)$, $e_2 = (0, 1)$.
Then $U = \langle e_1, e_2\rangle$.

The right cosets of H in G are:

$$\{1, AB\}, \{A, B\}, \{A^2, A^2B\}.$$

We can obtain a transversal by simply making a choice of representative from each right coset. Suppose we choose $Y = \{AB, B, A^2\}$.

We are going to induce ρ up to G and so we must define $g\rho^G$ for each $g \in G$. Let's just do it for g = A.

Now we take, as a basis for $\mathbb{C}Y \otimes U$ the 6 elements:

$$AB \otimes e_1, AB \otimes e_2, B \otimes e_1, B \otimes e_2, A^2 \otimes e_1, A^2 \otimes e_2.$$

Relative to this basis $g\rho^G$ will be represented as a 6×6 matrix. The *i*'th row will be the coefficients, relative to this basis of the image of the *i*'th basis vector under $g\rho^G$. Let's just find the 3rd row.

The third basis vector is $B \otimes e_1$, so we need to find the representative of the right coset in which Bg lies.

Now we're taking g = A, so:

$$Bg = BA = A^{-1}B = A^2B \in \{A^2, A^2B\}.$$

We chose A^2 as the representative of this coset.
So $BA \in HA^2$ and so $BA(A^2)^{-1} \in H$.
In fact $BA(A^2)^{-1} = BA^{-1} = AB$.
Yes, indeed this does belong to H and $(AB)\rho = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$.

Now $e_1(AB)\rho = (1, 1) = e_1 + e_2$.
Hence $(B \otimes e_1) = A^2 \otimes (e_1 + e_2)$

$$= A^2 \otimes e_1 + A^2 \otimes e_2.$$

These are the 5th and 6th basis vectors.

The corresponding coefficients are 1, with all other coefficients being zero.

Writing these coefficients as the third row of the matrix for $g\rho^G$ we get (0, 0, 0, 0, 1, 1).

Following the same method on the other rows we get:

$$A\rho^G = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Theorem 2: If ρ is a representation of $H \leq G$, ρ^G is a representation of G.

Proof: $(y_i \otimes u_j).(a\rho^G).(b\rho^G)$

$= (y_k \otimes u_j.(y_i a y_k^{-1})\rho).(b\rho^G)$ where $y_i a \in Hy_k$

$= y_h \otimes u_j(y_i a y_k^{-1})\rho.(y_k b y_h^{-1})\rho$ where $y_k b \in Hy_h$

$= y_h \otimes u_j(y_i a y_k^{-1} y_k b y_h^{-1})\rho$

$= y_h \otimes u_j(y_i a b y_h^{-1})\rho$.

Now $y_i(ab) \in Hy_h$ and so:

$$(y_i \otimes u_j).(ab)\rho^G = y_h \otimes u_j(y_i a b y_h^{-1})\rho.$$

Hence $(ab)\rho^G = (a\rho^G)(b\rho^G)$.

Let ρ^G be the representation of G induced from the representation $\rho: H \to \mathrm{End}_F(U)$ of the subgroup H.

Let $Y = \{y_1, \dots, y_n\}$ be a right transversal for H and let $\{u_1, \dots, u_m\}$ be a basis for U.
Then relative to the basis $\{y_1 \otimes u_1, y_1 \otimes u_2, \dots, y_n \otimes u_m\}$ for $FY \otimes U$, the matrix for $g\rho^G$ is the $mn \times mn$ matrix made up of $m \times m$ blocks (M_{ij}) with:

$$M_{ij} = \begin{cases} \text{the matrix of } (y_i g y_j^{-1})\rho \text{ if } y_i g y_j^{-1} \in H \\ 0 \text{ otherwise} \end{cases}.$$

There is exactly one non-zero block for each i and for each j. ✋☺

Theorem 3: If $H \le G$ and χ is a character of H and g lies in the conjugacy class Γ (of G) then

$$g\chi^G = \frac{|G|}{|H|} \times \frac{|\Gamma \cap H|}{|\Gamma|} \times \frac{\sum h\chi}{|\Gamma \cap H|}$$

where the sum is over all $h \in \Gamma \cap H$.

This is:

index of H in G × **proportion of Γ lying in H** × **average character for these elements**

Proof: The trace of $g\rho^G$ is the sum of the traces of the diagonal blocks. These blocks are of the form $(xgx^{-1})\rho$ where $xgx^{-1} \in H$.

Let $\Gamma^* = \Gamma \cap H$. Thus:

$g\chi^G = \sum(ygy^{-1})\chi$ where the sum is over those

$y \in Y$ where $ygy^{-1} \in H$.

$= \frac{1}{|H|}\sum(xgx^{-1})\chi$ where the sum is over those

$x \in G$ where $xgx^{-1} \in H$.

(This is because $x = hy \to y = h^{-1}x \to ygy^{-1} = h^{-1}xgx^{-1}h$)

$$= \frac{1}{|H|} \times |C_G(g)| \times \sum_{y \in \Gamma^*} y\chi$$

$$= \frac{1}{|H|} \times \frac{|G|}{|\Gamma|} \times \sum_{y \in \Gamma^*} y\chi$$

This is because $x_1gx_1^{-1} = x_2gx_2^{-1}$ if and only if

$x_2^{-1}x_1 \in C_G(g)$. ✋☺

$$= \frac{|G|}{|H|} \times \frac{|\Gamma^*|}{|\Gamma|} \times \frac{\sum_{y\in\Gamma^*} y\chi}{|\Gamma^*|}.$$

Example 3: In addition to S_3 being isomorphic to a quotient group of S_4 it is also a subgroup.

The character table, over ℂ, of S_3 is (swapping the 2nd and 3rd columns so that it corresponds to the order :

class	Γ_1	Γ_2	Γ_3
size	1	3	2
χ_1	1	1	1
χ_2	1	–1	1
χ_3	2	0	–1

The index of S_3 in S_4 is 24/6 = 4.
The proportions of Ω_1, Ω_2, Ω_3, Ω_4, Ω_5 that lie in S_3 are:

1, ½, ¼, 0, 0 respectively.

We now induce χ_1, χ_2, χ_3 to S_4.
Note that we have to be careful to make sure that we're using the correct columns of the subgroup's character table. Unless we organise it specially, as we did here, it need not be the case that the columns correspond in the order in which they're listed.

class	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
size	1	6	8	6	3
χ_1^G	4	2	1	0	0
χ_2^G	4	2	1	0	0
χ_3^G	8	0	–1	0	0

None of these induced characters are irreducible. We'll later see that χ_1^G and χ_2^G are each the sum of two irreducible characters while χ_3^G is the sum of three. However contained within them are the two missing irreducible characters that we didn't pick up when inducing from quotient groups.

When inducing up from quotient groups one should have the complete character table for the quotient available. Inducing up from the trivial character will only give the trivial character. On the other hand, inducing up from the trivial character of a subgroup will not just give the trivial character. The induced character may or may not be irreducible, but even if reducible it is often possible to subtract already obtained irreducible characters to obtain a new irreducible character. One should induce up the linear character from as many subgroups as possible before putting one's efforts into obtaining complete character tables for the subgroups.

§5.4. Inducing Down

Normally one would proceed from smaller groups, as subgroups or quotient groups, to obtain a new character table. However there are some instances where it's worth going from a larger group to a smaller. For example it's somewhat easier to construct the character table for S_n than the corresponding A_n and it's easier to find the character table for SL(n, p), the group of $n \times n$ matrices over $\mathbb{Z}_p$ that have determinant 1, rather than PSL(n, p), which is quotient of SL(n, p) by its centre.

Suppose χ is an irreducible character for G and that takes the value 1 on the elements of some normal subgroup H. Then H is in the kernel of the representation ρ that corresponds to χ and χ will be an irreducible character for G/H.

Example 4: The character table for G = S_4 is:

	I	(××)(××)	(×××)	(××)	(××××)
size	1	3	8	6	6
χ_1	1	1	1	1	1
χ_2	1	1	1	–1	–1
χ_3	2	2	–1	0	0
χ_4	3	–1	0	1	–1
χ_5	3	–1	0	–1	1

The normal subgroup H, consisting of the first two conjugacy classes, is in the kernel of ρ_1, ρ_2 and ρ_3. Hence χ_1, χ_2 and χ_3 give irreducible characters for G/H, which is isomorphic to S_3.

	I	(×××)	(××)
size	1	2	3
χ_1	1	1	1
χ_2	1	1	–1
χ_3	2	–1	0

If χ is an irreducible character of G then the restriction of χ to a subgroup H will give a character, not necessarily irreducible, of H.

Example 5: If G = S_4 and H = S_3, the subgroup consisting of permutations fixing the symbol 4, the irreducible characters of G give the following characters of H.

	I	(×××)	(××)
size	**1**	**2**	**3**
χ_1	1	1	1
χ_2	1	1	–1
χ_3	2	–1	0
χ_4	3	0	1
χ_5	3	0	–1

Here we get all the irreducible characters of H together with a couple of reducible ones:

$$\chi_4 = \chi_1 + \chi_3 \text{ and } \chi_5 = \chi_2 + \chi_3.$$

If H = A_4 we get the following characters of A_4.

	I	(××)(××)	(×××)	
size	**1**	**3**	**4**	**4**
χ_1	1	1	1	1
χ_2	1	1	1	1
χ_3	2	2	–1	–1
χ_4	3	–1	0	0
χ_5	3	–1	0	0

This gives two of the four irreducible characters of A_4. The character χ_3 is the sum of the two remaining characters of A_4 and we would need some extra work to find these.